*Journal of Statistical Physics, Vol. 23, No. 5, 1980*

# Differential Real-Space Renormalization of the $d$-Dimensional Gaussian Model

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*Received September 27, 1979*

With the aid of the differential real-space method we derive exact renormalization group (RG) equations for the Gaussian model in $d$ dimensions. The equations involve $d+1$ spatially dependent nearest-neighbor interactions. We locate a critical fixed point and obtain the exact thermal critical index $y_T = 2$. A special trajectory of the full nonlinear RG transformation is found and the free energy of the corresponding initial state calculated.

**KEY WORDS:** Real-space renormalization; differential renormalization; Gaussian model; free energy.

## 1. INTRODUCTION

It has recently been shown[1,2] that it is possible within the framework of real-space renormalization[4] to rescale a lattice system infinitesimally and thus obtain a renormalization group (RG) transformation in differential form. In this way exact RG equations have been derived for the two-dimensional Ising model. The fact that such equations exist is remarkable. It offers the possibility to study[1,2,4] for the first time the working of an exact RG in a nontrivial model. In order to further assess the applicability of the differential real-space renormalization, Van Saarloos *et al.*[5] have used it to study the Ising chain in a magnetic field, and Yamazaki *et al.*[6,7] have studied the two- and three-dimensional Gaussian models. A recent study of van der Waals spin systems by Dekeyser and Stella[8] is inspired by the same ideas.

Supported by Deutsche Forschungsgemeinschaft through Sonderforschungsbereich 130 Ferroelektrika.

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0022-4715/80/1100-0609$03.00/0 

In this work we consider the Gaussian model in $d$ dimensions. We extend the results obtained earlier[6,7] and present details of the calculations. Although the Gaussian model is trivial, its RG equations are not. By studying these for general $d$ we prepare the ground for subsequent investigation of non-Gaussian perturbations.

In two dimensions the differential RG equations were derived for a hexagonal (or, equivalently, triangular) lattice. In this paper (Section 2) we consider the appropriate generalization of a hexagonal lattice to $d$ dimensions, viz. a $(d+1)$-hedral lattice, in which each site has $d+1$ neighbors. As in the $d=2$ case, we work in the space of Hamiltonians with nearest-neighbor interactions that are spatially dependent on a thermodynamic scale. The RG transformation can then be expressed as a set of $d+1$ partial differential equations for these interactions [Eqs. (34)]. For $d>2$ we find that these equations have to be supplemented by a set of conditions [Eqs. (36)] which restrict the number of possible solutions.

In Section 3 we obtain a critical fixed-point solution of the RG transformation. We study the behavior of the linearized RG around the fixed point and obtain the exact thermal critical index $y_T = 2$. We also obtain a special solution of the full nonlinear RG equations. In Section 4 we derive an expression for the free energy as the trajectory integral of an explicitly given function. As an application we calculate the free energy for the special trajectory. The paper ends with a conclusion (Section 5).

## 2. DERIVATION OF THE RENORMALIZATION GROUP EQUATIONS

In this section we derive a differential renormalization group (RG) transformation for the Gaussian model by means of the real-space method. Section 2.1 summarizes the basic idea, which has been more fully exposed in Ref. 2. In Section 2.2 we describe the $d$-dimensional lattices that we shall consider. In Section 2.3 we derive a star-triangle transformation for such lattices. This transformation will play an essential role in what follows. In Section 2.4 we present the actual derivation of the RG equations. This section is concluded with a summary.

### 2.1. The Basic Idea

In general an RG transformation is constructed by mapping some original Hamiltonian $\mathscr{H}$, defined on a given lattice $\mathscr{L}$, onto a new Hamiltonian $\mathscr{H}'$ defined on a lattice $\mathscr{L}'$ which is identical to $\mathscr{L}$ except that it has a larger lattice constant. In the differential real-space method one takes for $\mathscr{L}$ and $\mathscr{L}'$ two large but finite lattices, defined in a suitably chosen spatial

domain. In Fig. 1a the two-dimensional case is shown: $\mathscr{L}$ and $\mathscr{L}'$ are triangular lattices confined to a triangularly shaped domain of side length $L$, and differ only in that $\mathscr{L}$ has one more lattice site along each side than $\mathscr{L}'$. The lattice constants are $a$ and $aL/(L-a)$, respectively. Clearly the lattice $\mathscr{L}'$ can be obtained from the lattice $\tilde{\mathscr{L}}$ shown in Fig. 1b by a uniform dilation in space.

It will be convenient to perform the mapping from $\mathscr{H}$ to $\mathscr{H}'$ in two steps: (i) we first transform $\mathscr{H}$ to a Hamiltonian $\tilde{\mathscr{H}}$ defined on $\tilde{\mathscr{L}}$; this step contains the essentials of the RG transformation; (ii) we obtain $\mathscr{H}'$ from $\tilde{\mathscr{H}}$ by a trivial dilation of the coordinate system.

## 2.2. The $d$-Dimensional Lattice

The three-dimensional lattice $\mathscr{L}$ can be obtained by layering up $L/a+1$ two-dimensional lattices of decreasing size along the $x_3$ axis, as in Fig. 2. The layer distance is chosen such that nearest neighbors in different layers are again at distance $a$. Thus in three dimensions $\mathscr{L}$ is a tetrahedral (i.e., fcc) lattice confined to a tetrahedrally shaped region of space with side of length $L$. In general the $d$-dimensional lattice $\mathscr{L}$ is obtained by layering up $L/a+1$ $(d-1)$-dimensional lattices of decreasing size along the $x_d$ axis. The result is a "$(d+1)$-hedral" lattice confined to a $(d+1)$-hedrally shaped region of $d$-dimensional space, with side of length $L$. The origin of the coordinate system is chosen in the center of the $(d+1)$-hedron.

The two interpenetrating lattices $\mathscr{L}$ and $\tilde{\mathscr{L}}$ form a hexagonal lattice for

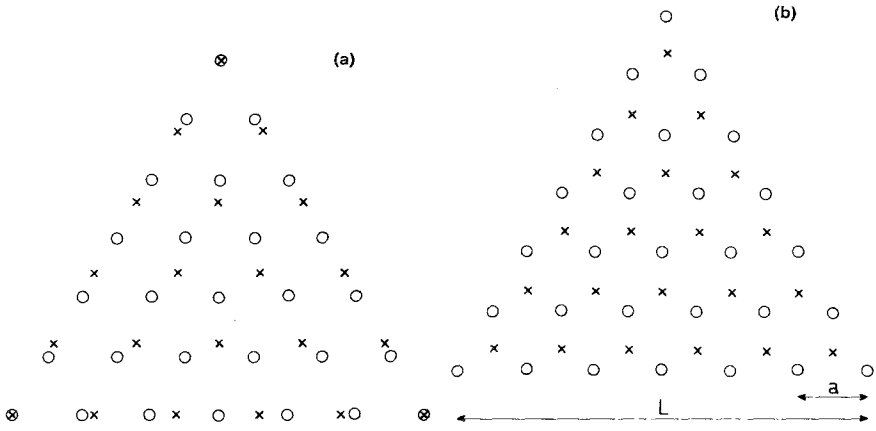


Fig. 1. The two-dimensional case. (a) The lattices $\mathscr{L}$ (circles) and $\mathscr{L}'$ (crosses) occupy the same spatial domain. Their lattice constants are $a$ and $aL/(L-a)$, respectively. (b) The lattice $\tilde{\mathscr{L}}$ (crosses) goes over into $\mathscr{L}'$ by a uniform dilation in space.

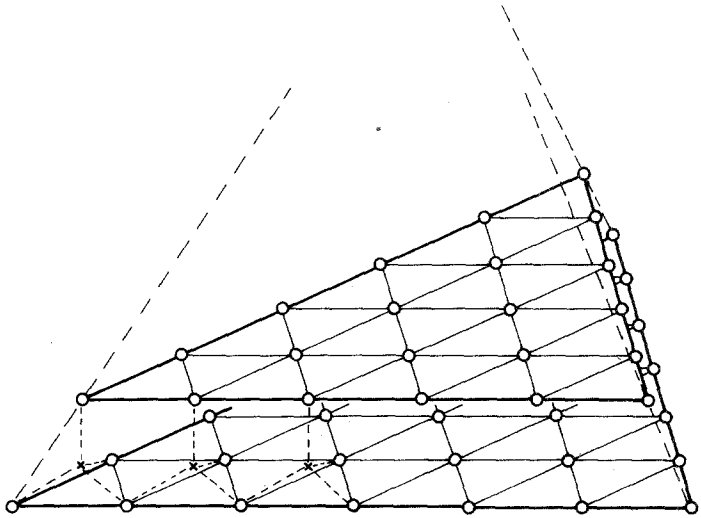

Fig. 2. The two-dimensional case. Two layers of the lattice $\mathscr{L}$ (circles) are shown, as well as a few sites of the lattice $\tilde{\mathscr{L}}$.

$d = 2$ (see Fig. 1b), a diamond lattice for $d = 3$, and a $d$-dimensional "hyper-hexagonal" lattice in the general case. In this lattice, each site of $\tilde{\mathscr{L}}$ has $d + 1$ nearest neighbors, all located on $\mathscr{L}$, and which are at the vertices of an elementary upward pointing $(d + 1)$-hedron of $\mathscr{L}$. Conversely each site of $\mathscr{L}$ is the center of an elementary downward pointing $(d + 1)$-hedron of $\tilde{\mathscr{L}}$. The vector distance from a site of $\mathscr{L}$ to a neighboring site of $\tilde{\mathscr{L}}$ takes $d + 1$ possible values that we denote $a\mathbf{e}_1, a\mathbf{e}_2, \ldots, a\mathbf{e}_{d+1}$. The case $d = 2$ is shown in Fig. 3. In the general case the vectors $\mathbf{e}_l$ are given by

$$\mathbf{e}_l = (e_{l1}, e_{l2}, \ldots, e_{ld}), \qquad l = 1, \ldots, d + 1 \tag{1}$$

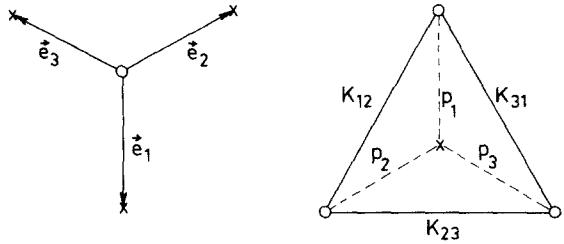


Fig. 3. The basis vectors $\mathbf{e}_i$, the interactions $K_{ij}$, and the couplings $p_i$, shown for the two-dimensional case.

where

$$\begin{aligned} e_{lm} &= 0, \qquad 1 \leqslant m \leqslant l-2 \\ e_{l,l-1} &= -(l-1)\alpha_{l-1} \\ e_{lm} &= \alpha_m, \qquad l \leqslant m \leqslant d \end{aligned} \tag{2}$$

and

$$\alpha_m = [2m(m+1)]^{-1/2} \tag{3}$$

We have the relations

$$\begin{aligned} \mathbf{e}_l \cdot \mathbf{e}_l &= d/[2(d+1)] \\ \mathbf{e}_l \cdot \mathbf{e}_m &= -1/[2(d+1)], \qquad l \neq m \\ \sum_{l=1}^{d+1} \mathbf{e}_l &= 0 \end{aligned} \tag{4}$$

## 2.3. A Star-Triangle Transformation for the Gaussian Model

The so-called star-triangle transformation for Ising models is well known and has been described in detail, e.g., by Syozi.[9] Here we derive an analogous transformation for Gaussian variables, which will play an essential role in what follows.

Consider a hyperhexagonal lattice (as defined in the previous subsection) of infinite size, whose sites $\mathbf{R}$ are occupied by Gaussian spins $S(\mathbf{R})$. Let there be an interaction of strength $p_l$ between any pair of nearest-neighbor spins a distance $a\mathbf{e}_l$ apart ($l = 1,\ldots, d+1$). The partition function of this system can be written

$$\begin{aligned} Z = &\int_{-\infty}^{\infty} \prod_{\mathbf{R}}^{\circ} \frac{dS(\mathbf{R})}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \prod_{\mathbf{R}}^{\times} \frac{dS(\mathbf{R})}{(2\pi)^{1/2}} \\ &\times \exp\left[ -\frac{1}{2} \sum_{\mathbf{R}}^{\circ,\times} S^2(\mathbf{R}) + \sum_{\mathbf{R}}^{\times} \sum_{i=1}^{d+1} p_i S(\mathbf{R} - a\mathbf{e}_i) S(\mathbf{R}) \right] \end{aligned} \tag{5}$$

where the indices $^{\circ}$ and $^{\times}$ distinguish the two sublattices (their sites being indicated by circles and crosses, respectively, in Figs. 1–3). If in Eq. (5) we carry out a partial trace on the spins of the sublattice $^{\times}$, and rescale the remaining spin variables such that in the end all self-interactions $S^2(\mathbf{R})$ occur again with coefficient $-\frac{1}{2}$, we find

$$\begin{aligned} Z = \zeta^{-1/2} &\int_{-\infty}^{\infty} \prod_{\mathbf{R}}^{\circ} \frac{dS(\mathbf{R})}{(2\pi)^{1/2}} \\ &\times \exp\left[ -\frac{1}{2} \sum_{\mathbf{R}}^{\circ} S^2(\mathbf{R}) + \sum_{\mathbf{R}}^{\times} \sum_{1 \leqslant i < j \leqslant d+1} K_{ij} S(\mathbf{R} - a\mathbf{e}_i) S(\mathbf{R} - a\mathbf{e}_j) \right] \end{aligned} \tag{6}$$

in which

$$K_{ij} = z^{-1} p_i p_j, \qquad 1 \leqslant i < j \leqslant d+1 \tag{7}$$

$$z = 1 - \sum_{l=1}^{d+1} p_l^2 \tag{8}$$

and $\zeta^{-1/2}$ is the Jacobian of the spin rescaling, determined by $\zeta = z^{N_\circ}$, $N_\circ$ being the number of sites in the sublattice $^\circ$. Equation (6) expresses $Z$ as the partition function of a $(d+1)$-hedral lattice with nearest-neighbor interactions $K_{ij}$. These interactions are not all independent, but in view of Eq. (7) satisfy the relations

$$K_{ij} K_{kl} = K_{ik} K_{jl} \tag{9}$$

where $i, j, k, l$ are all different.[5] It follows that the partition function of an initially given (homogeneous) nearest-neighbor Hamiltonian on a $(d+1)$-hedral lattice has a representation of type (5) only if its interaction parameters satisfy the relations (9). If such is the case, then one can obtain the $p_i$ from the $K_{ij}$ by inverting Eqs. (7) and (8), which leads to

$$p_i = z^{1/2} (K_{ij} K_{ik} / K_{jk})^{1/2} \tag{10}$$

$$z^{-1} = 1 + \sum_{l=1}^{d+1} (K_{lj} K_{lk} / K_{jk}) \tag{11}$$

Here the indices $j$ and $k$ are arbitrary [but, of course, $j, k \neq i$ in Eq. (10), $j, k \neq l$ in Eq. (11), and $j \neq k$].

It is sometimes profitable to replace the $\frac{1}{2}d(d+1)$ dependent variables $K_{ij}$ by the $d+1$ independent combinations

$$k_i = (K_{ij} K_{ik} / K_{jk})^{1/2}, \qquad i = 1, \ldots, d+1 \tag{12}$$

Equation (9) guarantees that this definition is independent of the indices $j$ and $k$. The star-triangle transformation reads, in terms of the variables $k_i$ and $p_j$,

$$k_i = p_i \Big/ \left(1 - \sum_l p_l^2\right)^{1/2} \tag{13}$$

and inversely

$$p_i = k_i \Big/ \left(1 + \sum_l k_l^2\right)^{1/2} \tag{14}$$

[5] The $\frac{1}{2}d(d+1)$ interactions $K_{ij}$ are parametrized by $d+1$ independent couplings $p_i$. Consequently, there can be at most $\frac{1}{2}d(d+1) - (d+1) = \frac{1}{2}(d-2)(d+1)$ independent relations of type (9). This may also be verified explicitly.

## 2.4. Derivation of the RG Equations

The RG equations are derived by generalizing the star-triangle transformation of the previous subsection to inhomogeneous Hamiltonians defined on the lattices $\mathscr{L}$ and $\tilde{\mathscr{L}}$ of Section 2.2. (The finite extent of these lattices implies certain boundary conditions that are considered later.)

We consider an initially given inhomogeneous Gaussian Hamiltonian $\mathscr{H}$ on the lattice $\mathscr{L}$,

$$\mathscr{H} = \sum_{\mathbf{R}}^{\times} \sum_{1 \leqslant i < j \leqslant d+1} K_{ij}(\mathbf{R}) S(\mathbf{R} - a\mathbf{e}_i) S(\mathbf{R} - a\mathbf{e}_j) - \frac{1}{2} \sum_{\mathbf{R}}^{\circ} S^2(\mathbf{R}) \tag{15}$$

The nearest-neighbor interactions $K_{ij}(\mathbf{R})$ include the temperature factor $-1/k_B T$; $\sum^{\times}$ indicates a summation through all sites of $\tilde{\mathscr{L}}$, and $\sum^{\circ}$ a summation through all sites of $\mathscr{L}$. We have adopted the convention of labeling each bond $K_{ij}$ by the coordinate of the center of the *up*-$(d+1)$-hedron of which it is an edge. The partition function of the system described by Eq. (15) can be expressed as

$$Z_{L/a}[K] = \int_{-\infty}^{\infty} \prod_{\mathbf{R}}^{\circ} \frac{dS(\mathbf{R})}{(2\pi)^{1/2}} e^{\mathscr{H}} \tag{16}$$

The subscript $L/a$ is a reminder of the linear dimension of the lattice, and $K$ stands for the entire set $\{K_{ij}(\mathbf{R})\}$.

We shall now assume that in every up-$(d+1)$-hedron of $\mathscr{L}$ the interactions $K_{ij}(\mathbf{R})$ are restricted by the relations (9), i.e.,

$$K_{ij}(\mathbf{R}) K_{kl}(\mathbf{R}) = K_{ik}(\mathbf{R}) K_{jl}(\mathbf{R}) \qquad \text{for all } \mathbf{R} \tag{17}$$

It is then possible to represent $Z_{L/a}[K]$ by the partition function of a "coupling" Hamiltonian $\mathscr{H}_c$ defined on the combined hyperhexagonal lattice $\mathscr{L} \cup \tilde{\mathscr{L}}$. To establish this we generalize the results of the previous subsection to the inhomogeneous lattice $\mathscr{L}$. We find

$$Z_{L/a}[K] = \zeta_{\circ}^{1/2} Z_{c,L/a}[p] \tag{18}$$

with

$$Z_{c,L/a}[p] = \int_{-\infty}^{\infty} \prod_{\mathbf{R}}^{\circ} \frac{dS(\mathbf{R})}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \prod_{\mathbf{R}}^{\times} \frac{dS(\mathbf{R})}{(2\pi)^{1/2}} e^{\mathscr{H}_c} \tag{19}$$

and

$$\mathscr{H}_c = \sum_{\mathbf{R}}^{\times} \sum_{i=1}^{d+1} p_i(\mathbf{R}) S(\mathbf{R}) S(\mathbf{R} - a\mathbf{e}_i) - \frac{1}{2} \sum_{\mathbf{R}}^{\circ,\times} S^2(\mathbf{R}) \tag{20}$$

in which the couplings $p_i(\mathbf{R})$ follow from the $K_{ij}(\mathbf{R})$ by

$$p_i(\mathbf{R}) = z_{\circ}^{1/2}(\mathbf{R} - a\mathbf{e}_i)[K_{ij}(\mathbf{R}) K_{ik}(\mathbf{R}) / K_{jk}(\mathbf{R})]^{1/2} \tag{21}$$

$$z_\circ(\mathbf{R}) = \left\{1 + \sum_{l=1}^{d+1} [K_{lj}(\mathbf{R} + a\mathbf{e}_l)K_{lk}(\mathbf{R} + a\mathbf{e}_l)/K_{jk}(\mathbf{R} + a\mathbf{e}_l)]\right\}^{-1} \tag{22a}$$

$$= 1 - \sum_{l=1}^{d+1} p_l^2(\mathbf{R} + a\mathbf{e}_l) \tag{22b}$$

The remark on the indices $j$ and $k$ following Eq. (11) applies here in the same way. Finally, in Eq. (18)

$$\zeta_\circ = \prod_{\mathbf{R}}^{\circ} z_\circ(\mathbf{R}) \tag{23}$$

This completes our description of the relation between $\mathscr{H}$ and $\mathscr{H}_c$.

We consider again Eq. (19). By interchanging the two sets of integrations and performing those on the spins of $\mathscr{L}$, we can express $z_{c,L}[p]$ as the partition function of a new Hamiltonian $\tilde{\mathscr{H}}$ defined on $\tilde{\mathscr{L}}$. To establish this we generalize the results of the previous subsection to the down-$(d+1)$-hedra of the lattice $\tilde{\mathscr{L}}$. We find

$$Z_{c,L/a}[p] = \zeta_\times^{-1/2} Z_{L/a-1}[\tilde{K}] \tag{24}$$

with

$$Z_{L/a-1}[\tilde{K}] = \int_{-\infty}^{\infty} \prod_{\mathbf{R}}^{\times} \frac{dS(\mathbf{R})}{(2\pi)^{1/2}} \exp(\tilde{\mathscr{H}}) \tag{25}$$

and

$$\tilde{\mathscr{H}} = \sum_{\mathbf{R}} \sum_{1 \leqslant i < j \leqslant d+1} \tilde{K}_{ij}(\mathbf{R}) S(\mathbf{R} - a\mathbf{e}_i) S(\mathbf{R} - a\mathbf{e}_j) - \frac{1}{2} \sum_{\mathbf{R}}^{\times} S^2(\mathbf{R}) \tag{26}$$

The first summation in Eq. (26) runs through the centers of all up-$(d+1)$-hedra in $\tilde{\mathscr{L}}$, and the interactions $\tilde{K}_{ij}$ follow from the $p_i$ by

$$\tilde{K}_{ij}(\mathbf{R}) = z_\times^{-1/2}(\mathbf{R} - a\mathbf{e}_i) z_\times^{-1/2}(\mathbf{R} - a\mathbf{e}_j) p_i(\mathbf{R} - a\mathbf{e}_j) p_j(\mathbf{R} - a\mathbf{e}_i) \tag{27}$$

$$z_\times(\mathbf{R}) = 1 - \sum_{l=1}^{d+1} p_l^2(\mathbf{R}) \tag{28}$$

and finally in Eq. (24)

$$\zeta_\times = \prod_{\mathbf{R}}^{\times} z_\times(\mathbf{R}) \tag{29}$$

Thus we have determined the relation between $\mathscr{H}_c$ and $\mathscr{H}$.

Lastly, a uniform dilation of the coordinate system maps a point $\mathbf{R} \in \mathscr{L}$ onto a point $\mathbf{R}' \in \mathscr{L}'$ (see Fig. 1a) and takes us from $\tilde{\mathscr{H}}$ to $\mathscr{H}'$. We define

$$\mathbf{R}' = \mathbf{R}[L/(L-a)] \tag{30a}$$

$$K'_{ij}(\mathbf{R}') = \tilde{K}_{ij}([(L-a)/L]\mathbf{R}') = \tilde{K}_{ij}(\mathbf{R}) \tag{30b}$$

Hence $\mathscr{H}'$ is obtained from $\tilde{\mathscr{H}}$ by a mere relabeling of the spin variables and interaction parameters, which does not affect their values. In particular,

$$Z_{L/a-1}[\tilde{K}] = Z_{L/a-1}[K'] \tag{31}$$

We can now combine the above sequence of mappings $\mathscr{H} \to \mathscr{H}_c \to \tilde{\mathscr{H}} \to \mathscr{H}'$ to obtain a single RG transformation $\mathscr{H} \to \mathscr{H}'$. An RG trajectory in Hamiltonian space is then constructed by iterating according to $\mathscr{H} \to \mathscr{H}' \to \mathscr{H}'' \to \ldots$. The transformation $\mathscr{H} \to \mathscr{H}'$, however, has been realized only under the restrictive conditions (17) on $\mathscr{H}$. Hence in order to be able to iterate we have to verify that the interactions $K'_{ij}(\mathbf{R})$ [or $\tilde{K}_{ij}(\mathbf{R})$] satisfy the same restrictions (17) in every up-$(d+1)$-hedron of $\mathscr{L}'$ [or $\tilde{\mathscr{L}}$]. Upon employing Eq. (27), we see that this will be true if

$$\begin{aligned} &p_i(\mathbf{R} - a\mathbf{e}_j)p_j(\mathbf{R} - a\mathbf{e}_i)p_k(\mathbf{R} - a\mathbf{e}_l)p_l(\mathbf{R} - a\mathbf{e}_k) \\ &\quad = p_i(\mathbf{R} - a\mathbf{e}_k)p_k(\mathbf{R} - a\mathbf{e}_i)p_j(\mathbf{R} - a\mathbf{e}_l)p_l(\mathbf{R} - a\mathbf{e}_j) \end{aligned} \tag{32}$$

We shall now sketch how the RG equations can be obtained in explicit form. Details of the calculations are given in an appendix. We assume that the functions $K_{ij}(\mathbf{R})$ vary only on a scale $L \gg a$, so that we may consider them as continuous functions with gradients of order $1/L$. One can then combine Eqs. (30b), (27), (28), (21), and (22) to obtain an expression for $K'_{ij}(\mathbf{R})$ entirely in terms of the set $\{K_{ij}(\mathbf{R})\}$. After Taylor expansion it appears that $\delta K_{ij} \equiv K'_{ij} - K_{ij}$ is of order $a/L$. Setting $\mathbf{r} \equiv \mathbf{R}/L$ and $\delta t \equiv a/L$, one obtains in the limit $a/L \to 0$ an expression for $\partial K_{ij}(\mathbf{r}, t)/\partial t$. The detailed calculations are done in Appendix A, where it turns out to be profitable to work with the $d+1$ independent variables $k_i$ [see Eq. (13)].

The result is the differential RG equation

$$\frac{\partial k_i(\mathbf{r}, t)}{\partial t} = \sum_j \mathbf{B}_{ij}(k)\cdot\nabla k_j - \mathbf{r}\cdot\nabla k_i, \qquad i = 1,\ldots, d+1 \tag{33}$$

with $\mathbf{B}_{ij}(k)$ given by Eq. (A7), and where $\nabla$ stands for $\partial/\partial\mathbf{r}$. The term $-\mathbf{r}\cdot\nabla k_i$ is obviously due to the lattice dilation, Eq. (30b). The RG equation (33) can alternatively be expressed, either via Eq. (14) or by direct derivation (see Appendix A), in terms of the variables $p_i$. The result is that we find

$$\frac{\partial p_i(\mathbf{r}, t)}{\partial t} = \sum_j \mathbf{D}_{ij}(p)\cdot\nabla p_j - \mathbf{r}\cdot\nabla p_i, \qquad i = 1,\ldots, d+1 \tag{34}$$

with

$$\begin{aligned} \mathbf{D}_{ij}(p) = \frac{1}{d(d-1)}\frac{p_i}{p_j}\Bigg\{ & d\sum_k p_k{}^2\mathbf{e}_k + d(2\delta_{ij} - 1)\mathbf{e}_i \\ & + \left[\sum_k p_k{}^2 - d(d+1)p_j{}^2 - 1\right]\mathbf{e}_j\Bigg\} \end{aligned} \tag{35}$$

To complete our description of the RG transformation we also cast Eq. (32) in differential form, which yields the supplementary conditions

$$(\mathbf{e}_j - \mathbf{e}_k)\cdot\left(\frac{\nabla p_i}{p_i} - \frac{\nabla p_l}{p_l}\right) + (\mathbf{e}_i - \mathbf{e}_l)\cdot\left(\frac{\nabla p_j}{p_j} - \frac{\nabla p_k}{p_k}\right) = 0 \tag{36}$$

($i, j, k, l$ all different), to be obeyed by any solution of Eqs. (34) and (35). Finally we have to provide boundary conditions. As for the Ising model,[1,2] these follow by requiring that the star-triangle transformation from $p$ to $\tilde{K}$ also hold true along the border of the lattice, where one "leg" of the star of $p$-bonds is missing. This gives the boundary condition

$$p_i(\mathbf{r}) = 0 \qquad \text{for } \mathbf{r}\cdot\mathbf{e}_i = \tfrac{1}{2}(d+1)^{-1} \tag{37}$$

i.e., on the $i$th hyperplane bordering the lattice.

In summary, we have obtained a real-space RG transformation consisting of the $d+1$ partial differential equations (34), (35) for the functions $p_i(\mathbf{r}, t)$. The equations have to be solved for given $p_i(\mathbf{r}, 0)$ in a $d$-dimensional domain of $(d+1)$-hedral shape with edges of unit length and center in the origin. The boundary conditions (37) have to be imposed, and only those solutions that satisfy the supplementary conditions (36) are acceptable as true RG trajectories.

## 3. ANALYSIS OF THE RG TRANSFORMATION

In this section we analyze the RG transformation, Eqs. (34)–(37).

### 3.1. A Fixed Point of the RG Transformation

Earlier work on the Ising model suggests that we look for a fixed-point solution $[p_1^*(\mathbf{r}),\ldots, p_{d+1}^*(\mathbf{r})]$ of Eqs. (34) and (35) which is linear in the spatial coordinate. By explicit verification one can prove that indeed such a solution exists and is given by

$$p_i^*(\mathbf{r}) = [1/(d+1)] - 2\mathbf{r}\cdot\mathbf{e}_i, \qquad i = 1,\ldots, d+1 \tag{38}$$

Equation (38) satisfies the boundary conditions (37) as well as the supplementary conditions (36), and is therefore a fixed-point solution of the RG transformation. It represents a particular spatially inhomogeneous hyperhexagonal lattice, which is isotropic in the center and increasingly anisotropic toward the borders of the domain. The corresponding $(d+1)$-hedral fixed-point solution $K_{ij}^*(\mathbf{r})$ is easily obtained from it via Eq. (7). By symmetry one can see that Eq. (38) is in fact part of a set of $2^{d+1}$ fixed-point solutions, viz. $[\sigma_1 p_1^*(\mathbf{r}),\ldots, \sigma_{d+1} p_{d+1}^*(\mathbf{r})]$, where $\sigma_l = \pm 1$. On the $(d+1)$-hedral lattice $K_{ij}^*(\mathbf{r})$ is replaced correspondingly by $\sigma_i\sigma_j K_{ij}^*(\mathbf{r})$. Whereas the fixed point (38)

is ferromagnetic, the other fixed points represent various kinds of antiferromagnetic lattices.

### 3.2. An Invariant Subspace of Eqs. (34) and (35)

The fixed point (38) lies in the subspace $\mathscr{C}$ of $(d+1)$-plets $(p_1,\ldots,p_{d+1})$ that satisfy

$$p_1(\mathbf{r}) + p_2(\mathbf{r}) + \cdots + p_{d+1}(\mathbf{r}) = 1 \tag{39}$$

By straightforward but tedious algebra one can show that the flow generated by Eqs. (34) and (35) leaves this subspace invariant, i.e., that under the hypothesis (39) we have $\partial[\sum_l p_l(\mathbf{r}, t)]/\partial t = 0$. We remark that the relation $\sum_l p_l = 1$ is exactly the criticality condition for a homogeneous hyperhexagonal lattice[6] with couplings $p_1,\ldots,p_{d+1}$. We therefore identify the subspace $\mathscr{C}$ defined by Eq. (39) as the critical subspace of the RG transformation [strictly speaking this name applies only to those trajectories in $\mathscr{C}$ that satisfy Eq. (38)]. For the special case $d = 2$ the RG equations *in* the critical subspace can be shown[6] to be identical (up to a change of variables) to the analogous equations for the Ising model, which in turn were studied in detail by Knops and Hilhorst.[4]

### 3.3. A Temperature-Like Solution of the RG Transformation

Outside of the critical subspace we try to find a special solution of the RG transformation by making the ansatz

$$p_i(\mathbf{r}, t) = \beta(t) p_i^*(\mathbf{r}) \tag{40}$$

with $p_i^*(\mathbf{r})$ given by Eq. (38). This expression automatically satisfies Eqs. (36) and (37). By substitution into Eq. (34) and use of Eq. (35) we find that it is indeed a solution of the RG transformation provided that $\beta(t)$ satisfies

$$d\beta/dt = -\beta(1 - \beta^2) \tag{41}$$

Solving this equation for given $\beta(0) = \beta_0$, we find

$$\beta(t) = \beta_0 e^{-t}(1 - \beta_0^2 + \beta_0^2 e^{-2t})^{-1/2} \tag{42}$$

For $\beta_0 < 1$ the solution (40) decays, therefore, to the trivial infinite-temperature fixed point $p_i(\mathbf{r}) = 0$; for $\beta_0 > 1$ it reaches the value $p_i(\mathbf{r}) = \infty$ for finite $t$, namely $t = -\frac{1}{2}\ln(1 - \beta_0^{-2})$.

The singular critical behavior of the physical system is related to the eigenvalues of the linearized RG transformation. Linearizing Eq. (4) around the critical point $\beta = 1$, we see that it has an eigenvalue $y_T = 2$. This leads

[6] The properties of the homogeneous lattice are most easily obtained by Fourier transformation.

to the classical critical exponent $\nu = 1/y_T = \frac{1}{2}$ for the correlation functions, in agreement with the exact result.

We conclude this section by remarking that one may show, along the lines of Ref. 7, that the eigenvalue $y_T$ is in fact infinitely degenerate in the temperature subspace. We do not know, however, whether there exist any other eigensolutions, besides the one found above, that satisfy the supplementary conditions (36).

## 4. THE FREE ENERGY

### 4.1. Derivation of the Trajectory Integral for the Free Energy

Quite generally a differential RG equation leads to an expression for the free energy as an integral along an RG trajectory. We derive here the formulas appropriate to our case.

From Eqs. (18), (24), and (31) we find that the partition functions of the Hamiltonians $\mathscr{H}$ and $\mathscr{H}'$ are related by

$$Z_{L/a}[K] = \zeta_\circ^{1/2}\zeta_\times^{-1/2} Z_{L/a-1}[K'] \tag{43}$$

The (reduced) free energy $F_{L/a}[K] \equiv \ln Z_{L/a}[K]$ therefore satisfies the recursion relation

$$F_{L/a}[K] - F_{L/a-1}[K'] = (a/L) G_{L/a}[K] \tag{44}$$

where

$$\frac{a}{L} G_{L/a}[K] = \frac{1}{2}\left[\sum_{\mathbf{R}}^{\circ} \ln z_\circ(\mathbf{R}) - \sum_{\mathbf{R}}^{\times} \ln z_\times(\mathbf{R})\right] \tag{45}$$

Here we have used Eqs. (23) and (29) and anticipated that the rhs of Eq. (45) will be of order $a/L$.

In the limit $a/L \to 0$ the interactions $K_{ij}(\mathbf{R})$ converge to continuous functions which, after rescaling coordinates, we denoted as $K_{ij}(\mathbf{r})$. We assume here the asymptotic expansions

$$F_{L/a}[K] = \frac{1}{d!}\left(\frac{L}{a}\right)^d f[K(\mathbf{r})] + O\left[\left(\frac{L}{a}\right)^{d-1}\right] \tag{46a}$$

$$G_{L/a}[K] = \frac{1}{d!}\left(\frac{L}{a}\right)^d c[K(\mathbf{r})] + O\left[\left(\frac{L}{a}\right)^{d-1}\right] \tag{46b}$$

Here $f$ is the free energy per site averaged over the $(d+1)$-hedral domain. Substitution of Eq. (46) into Eq. (44) leads, for $a/L \to 0$, to the differential equation

$$\partial f/\partial t = df - c \tag{47}$$

The solution

$$f[K(\mathbf{r}, t)] = \int_t^\infty e^{d(t-\tau)}\{c[K(\mathbf{r}, \tau)] - df[K(\mathbf{r}, \infty)]\}\, d\tau + f[K(\mathbf{r}, \infty)] \quad (48)$$

reduces for $t = 0$ to the free energy of the initially given system.

It remains to derive from Eqs. (45) and (46b) an explicit expression for the functional $c[K]$. In Appendix B we show that this leads to

$$c[K(\mathbf{r})] = \frac{1}{V_d} \int_\Delta \left( dg + \mathbf{r} \cdot \nabla g + \sum_l \mathbf{e}_l \cdot \nabla p_l \frac{\partial g}{\partial p_l} \right) d\mathbf{r} \quad (49)$$

where the integral is on the $(d + 1)$-hedral domain and $V_d$ is the $d$-dimensional volume of this domain, and where

$$g(p_1, \ldots, p_{d+1}) = \frac{1}{2} \ln\left(1 - \sum_{l=1}^{d+1} p_l^2\right) \quad (50)$$

This completes the results of this subsection: once for a given initial system the trajectory $K_{ij}(\mathbf{r}, t)$ has been calculated from the RG equations, its free energy follows from Eqs. (48)–(50).

## 4.2. Application

We calculate by the above method the free energy of the special inhomogeneous system with couplings

$$K_{ij}^{\mathrm{sp}}(\mathbf{r}) = \beta_0^2 p_i^*(\mathbf{r}) p_j^*(\mathbf{r}) \Big/ \left[1 - \beta_0^2 \sum_{l=1}^{d+1} p_l^{*2}(\mathbf{r})\right] \quad (51)$$

with $p_i^*(\mathbf{r})$ given by Eq. (38). The trajectory of this initial state, known from Section 3.3, can be described by the single $t$-dependent parameter $\beta(t)$. Upon using Eqs. (38) and (50) in Eq. (49), we find that we can write

$$c[K^{\mathrm{sp}}(\mathbf{r}, t)] = \frac{1}{V_d} \int_\Delta \frac{1}{2} \left[ d \ln\left(1 - \beta^2 \sum_l p_l^{*2}\right) + \beta^2 \left(1 - \sum_l p_l^{*2}\right)\left(1 - \beta^2 \sum_l p_l^{*2}\right)^{-1} \right] d\mathbf{r} \quad (52)$$

Since $K_{ij}^{\mathrm{sp}}(\mathbf{r}, \infty) = 0$, we find from Eqs. (52) and (48)

$$f[K^{\mathrm{sp}}(\mathbf{r})] = \frac{1}{V_d} \int_\Delta \int_0^\infty e^{-dt} \frac{1}{2} \left[ d \ln\left(1 - \beta^2 \sum_l p_l^{*2}\right) + \beta^2 \left(1 - \sum_l p_l^{*2}\right)\left(1 - \beta^2 \sum_l p_l^{*2}\right)^{-1} \right] dt\, d\mathbf{r} \quad (53)$$

We transform to the new variable of integration $x \equiv e^{-2t}$. Upon using the explicit expression (42) for $\beta(t)$ and once applying a partial integration to the term with the logarithm in Eq. (53), we find

$$f[K^{\mathrm{sp}}(\mathbf{r})] = \frac{1}{V_d} \int_\Delta \frac{1}{2} \ln\left[1 - {\beta_0}^2 \sum_l p_l^{*2}(\mathbf{r})\right] d\mathbf{r} + \frac{1}{2} {\beta_0}^2 \int_0^1 \frac{x^{d/2}}{1 - {\beta_0}^2 + {\beta_0}^2 x}\, dx \tag{54}$$

Finally we compare the result (54) with the solution obtained by the standard method of calculating the free energy by diagonalizing the Hamiltonian. For the (reduced) free energy per site of a $(d+1)$-hedral lattice with homogeneous couplings $K_{ij}$ this method yields

$$f(\{K_{ij}\}) = -\frac{1}{(2\pi)^{d+1}} \int_0^{2\pi} d\omega_1 \cdots \int_0^{2\pi} d\omega_{d+1} \times \frac{1}{2} \ln\left[1 - 2 \sum_{1 \le i < j \le d+1} K_{ij} \cos(\omega_i - \omega_j)\right] \tag{55}$$

We expect to obtain an alternative expression for $f[K^{\mathrm{sp}}(\mathbf{r})]$ if in Eq. (55) we take $K_{ij} = K_{ij}^{\mathrm{sp}}(\mathbf{r})$ and average on the domain. The result, slightly rewritten, is

$$\begin{aligned} f[K^{\mathrm{sp}}(\mathbf{r})] = {} & \frac{1}{V_d} \int_\Delta \frac{1}{2} \ln\left[1 - {\beta_0}^2 \sum_l p_l^{*2}(\mathbf{r})\right] d\mathbf{r} \\ & - \frac{1}{V_d} \int_\Delta d\mathbf{r} \frac{1}{(2\pi)^{d+1}} \int_0^{2\pi} d\omega_1 \cdots \int_0^{2\pi} d\omega_{d+1} \\ & \times \frac{1}{2} \ln\left\{1 - {\beta_0}^2 \left[\sum_{l=1}^{d+1} p_l^*(\mathbf{r}) \cos \omega_l\right]^2 - {\beta_0}^2 \left[\sum_{l=1}^{d+1} p_l^*(\mathbf{r}) \sin \omega_l\right]^2\right\} \end{aligned} \tag{56}$$

The last terms both in Eq. (54) and in Eq. (56) represent the average free energy (per cell of two spins) of the inhomogeneous hyperhexagonal lattice with couplings $\beta_0 p_i^*(\mathbf{r})$. We have verified that the two expressions are identical for $d = 1$, and that their high-temperature expansions agree to order ${\beta_0}^4$ for general $d$.

## 5. CONCLUSION

We have derived exact RG equations for the $d$-dimensional Gaussian model with nearest-neighbor interactions only. We considered the model on a special $(d+1)$-coordinated lattice. The RG transformation takes the form of a set of $d+1$ partial differential equations for the nearest-neighbor

couplings, which are position dependent. This spatial inhomogeneity is an essential feature of the differential position-space method, as discussed extensively earlier.[2] While being similar in spirit, our present investigation has gone beyond the study of the Ising model of Ref. 2 in two respects.

(i) We were able to carry through our method in arbitrary dimension $d$. For $d > 2$ one finds that there are supplementary conditions to be satisfied by any solution of the actual RG equations. The role of these conditions is to keep the interactions constrained to the nearest-neighbor type. It has been shown by van Leeuwen[10] how our equations can be embedded in a larger class of transformations involving arbitrary pair interactions. Since $d$ is arbitrary, our calculation lays the basis for perturbative treatment of non-Gaussian terms.[7] One can show, in fact, that $d = 4$ is a marginal dimension, as it should be.

(ii) Unlike in the Ising case, we could obtain a special explicit solution of the full nonlinear equations, corresponding to a trajectory from the critical to the high-temperature fixed point.[8] We calculated the free energy for an arbitrary state on this trajectory by evaluating the RG trajectory integral, and verified that the expression thus obtained agrees with the result from traditional methods.

## APPENDIX A. DERIVATION OF EQS. (33)–(35)

In order to obtain Eq. (33) and find the explicit form of $\mathbf{B}(k)$, we first express $\tilde{k}_i(\mathbf{R})$ in the set $\{k_i(\mathbf{R})\}$. From the definition, Eq. (12), where now all quantities have the argument $\mathbf{R}$, and from Eqs. (27) and (21) we obtain easily

$$\tilde{k}_i^2(\mathbf{R}) = A_i(\mathbf{R} - a\mathbf{e}_i)B_i(\mathbf{R}) \tag{A1}$$

with

$$A_i(\mathbf{R} - a\mathbf{e}_i) = \frac{z_\circ(\mathbf{R} - a\mathbf{e}_i - a\mathbf{e}_j)z_\circ(\mathbf{R} - a\mathbf{e}_i - a\mathbf{e}_k)}{z_\circ(\mathbf{R} - a\mathbf{e}_j - a\mathbf{e}_k)z_x(\mathbf{R} - a\mathbf{e}_i)]} \tag{A2}$$

$$B_i(\mathbf{R}) = \frac{k_i(\mathbf{R} - a\mathbf{e}_j)k_i(\mathbf{R} - a\mathbf{e}_k)k_j(\mathbf{R} - a\mathbf{e}_i)k_k(\mathbf{R} - a\mathbf{e}_i)}{k_j(\mathbf{R} - a\mathbf{e}_k)k_k(\mathbf{R} - a\mathbf{e}_j)} \tag{A3}$$

In Eq. (A2) we use Eqs. (28), (21), and (22) to get an expression entirely in terms of the $k_i(\mathbf{R})$. On Taylor expanding both this expression and Eq. (A3),

[7] Such a calculation was done by Kadanoff *et al.*,[11] who used a *discrete* RG transformation combined with a bond shifting approximation.

[8] The solutions of the nonlinear Ising RG equations found in Ref. 4 are not temperature-like, but lie in the critical subspace.

we find

$$A_i(\mathbf{R} - a\mathbf{e}_i) = 1 + \left(1 + \sum_m k_m{}^2\right)^{-1} \times \sum_l \left[(a\mathbf{e}_i - a\mathbf{e}_l) + \sum_n (a\mathbf{e}_n - a\mathbf{e}_l)k_n{}^2\right] \cdot \nabla k_l{}^2 + O\left(\frac{a^2}{L^2}\right) \quad \text{(A4)}$$

$$B_i(\mathbf{R}) = k_i{}^2(\mathbf{R})\left[1 - (a\mathbf{e}_j + a\mathbf{e}_k)\cdot\frac{\nabla k_i}{k_i} + (a\mathbf{e}_k - a\mathbf{e}_i)\cdot\frac{\nabla k_j}{k_j} + (a\mathbf{e}_j - a\mathbf{e}_i)\cdot\frac{\nabla k_k}{k_k}\right] + O\left(\frac{a^2}{L^2}\right) \quad \text{(A5)}$$

Where the coordinate has been suppressed it is equal to $\mathbf{R}$, and $\nabla \equiv \partial/\partial\mathbf{R}$. With the aid of Eqs. (30a) and (30b), we find the relation between $k_i'(\mathbf{R})$ and $\tilde{k}_i(\mathbf{R})$ to be

$$k_i'(\mathbf{R}) = \tilde{k}_i(\mathbf{R}) - \mathbf{R}\cdot\nabla k_i + O(a^2/L^2) \quad \text{(A6)}$$

Substituting Eqs. (A4)–(A6) into Eq. (A1) we obtain an expression for $k_i'^2(\mathbf{R}) - k_i{}^2(\mathbf{R})$. The expression still depends on the arbitrary indices $j$ and $k$, but we can symmetrize it by applying the operation $[d(d-1)]^{-1}\sum_{j\neq i}\sum_{k\neq i,j}$. Setting $\mathbf{r} \equiv \mathbf{R}/L$, $\delta k_i \equiv k_i' - k_i$, and $\delta t \equiv a/L$, we then find in the limit $a/L \to 0$ Eq. (33) with $\mathbf{B}_{ij}(k)$ given by

$$\mathbf{B}_{ij}(k) = \left(1 + \sum_l k_l{}^2\right)^{-1} k_i k_j \left[\mathbf{e}_i - \mathbf{e}_j + \sum_k k_k{}^2(\mathbf{e}_k - \mathbf{e}_j)\right] + (d-1)^{-1}k_i k_j^{-1}[(2\delta_{ij} - 1)\mathbf{e}_i - d^{-1}\mathbf{e}_j] \quad \text{(A7)}$$

To obtain Eqs. (34) and (35) one proceeds in an analogous fashion, starting from Eqs. (27) and (28) and the analogs of Eqs. (21) and (22) obtained by providing all bond parameters with a tilde. Since lattice rescaling commutes with the star-triangle transformation, one can first establish the relation between $\tilde{p}_i(\mathbf{R})$ and the $p_i(\mathbf{R})$, and then use that

$$p_i'(\mathbf{R}) = \tilde{p}_i(\mathbf{R}) - \mathbf{R}\cdot\nabla p_i + O(a^2/L^2)$$

## APPENDIX B. THE FUNCTIONAL $c[K]$

We derive an explicit expression for the functional $c[K]$ defined by Eqs. (45) and (46b). Since the sums in Eq. (45) cancel to leading order, we rewrite them as a single summation with summand of order $a/L$. To this end we define for each crossed site $\mathbf{R}$ a set of $d+1$ fractions $f_i(\mathbf{R})$ such that $\sum_{i=1}^{d+1} f_i(\mathbf{R}) = 1$, and split the sum on the crossed sites up accordingly into $d+1$ fractional

sums:

$$\frac{a}{L} G_{L/a}[K] = \frac{1}{2}\left[\sum_{\mathbf{R}}^{\circ} \ln z_{\circ}(\mathbf{R}) - \sum_{\mathbf{R}}^{\times} \sum_{l} f_l(\mathbf{R}) \ln z_{\times}(\mathbf{R})\right] \tag{B1}$$

$$= \frac{1}{2}\sum_{\mathbf{R}}^{\circ}\left[\ln z_{\circ}(\mathbf{R}) - \sum_{l} f_l(\mathbf{R} + a\mathbf{e}_l) \ln z_{\times}(\mathbf{R} + a\mathbf{e}_l)\right] \tag{B2}$$

The second equality holds provided that $f_l(\mathbf{R} + a\mathbf{e}_l) = 0$ for $\mathbf{R}$ in the $l$th border plane of the lattice. A convenient special choice for the $f_i(\mathbf{R})$ satisfying this requirement is

$$f_i(\mathbf{R}) = 1/(d+1) - 2(1 - ad/L)\mathbf{R}\cdot\mathbf{e}_i/L \tag{B3}$$

It is now straightforward to use Eqs. (B3), (22b), and (28) in Eq. (B2). After Taylor expanding and taking the limit $a/L \to 0$ one obtains Eq. (49).

## ACKNOWLEDGMENTS

The authors would like to thank Prof. J. M. J. van Leeuwen for fruitful discussions. One of us (YY) would like to thank Profs. M. Suzuki, K. Kawasaki, S. Katsura, S. Inawashiro, P. Szepfalusy, and A. Holz for valuable and stimulating discussions.

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